

Mirror symmetry, Langlands duality, and commuting elements of Lie groups

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Abstract. By normalizing a component of the space of commuting pairs of elements in a reductive Lie group G , and the corresponding space for the Langlands dual group, we construct pairs of hyperkähler orbifolds which satisfy the conditions to be mirror partners in the sense of Strominger-Yau-Zaslow. The same holds true for commuting quadruples in a compact Lie group. The Hodge numbers of the mirror partners, or more precisely their orbifold E -polynomials, are shown to agree, as predicted by mirror symmetry. These polynomials are explicitly calculated when G is a quotient of $\mathrm{SL}(n)$.

Mirror symmetry made its first appearance in 1990 as an equivalence between two linear sigma-models in superstring theory [10, 21]. The targets were Calabi-Yau 3-folds, so mirror symmetry predicted that these should come in pairs, M and \hat{M} , satisfying $h^{p,q}(M) = H^{p,3-q}(\hat{M})$.

Although many examples were known, the physics did not immediately provide any general construction of a mathematical nature for the mirror. Since then, however, two mathematical constructions have emerged: that of Batyrev [1] and Batyrev-Borisov [2], generalizing the original idea of Greene-Plesser [21], and that of Strominger-Yau-Zaslow [46] with which this paper is concerned.

Of the two, Batyrev's construction has the advantage of being precise, and more amenable to explicit calculations. One can prove, for example, that the Hodge numbers of the Batyrev mirror satisfy the desired relationship. On the other hand, it is deeply rooted in toric geometry. This has led skeptics to suggest that mirror symmetry is an intrinsically toric phenomenon, despite work [3, 40] extending Batyrev's point of view some ways beyond the toric setting.

The construction proposed by Strominger-Yau-Zaslow in 1996 has quite a different flavor. It is directly inspired by a physical duality, the so-called T -duality between sigma-models whose targets are dual tori. Remarkably, although it is supposed to transform one projective variety into another, the construction is not algebraic, or even Kähler, in nature. Rather, it is symplectic: one must find a foliation of M by special Lagrangian tori, and replace each torus with its dual.

This bold idea has already led to some interesting work on the existence of families of Lagrangian tori in Calabi-Yau 3-folds [22, 23, 41], which is essentially a problem in symplectic topology. But it is not yet sufficiently advanced that the mirror can be constructed in any precise sense, nor any of its invariants computed beyond the Euler characteristic. It is not even known how to construct families of tori which are special Lagrangian (as opposed to just Lagrangian). And the further questions of what complex structure to place on the dual family, and how to deal with singular fibers, remain mysterious.

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This paper will study a case where these formidable difficulties can be completely circumvented. The Strominger-Yau-Zaslow construction works like a charm, and the equality of the Hodge numbers can be completely verified. Furthermore, there is a suggestive relationship with the Langlands duality between reductive Lie groups. On the other hand, this case is in some ways quite distant from the original case of Calabi-Yau 3-folds. The spaces in question are singular and often non-compact. Moreover, their complex dimension is even and typically rather large.

To find the special Lagrangian tori, we exploit the existence of a *hyperkähler* structure. This involves three complex structures on M which satisfy the commutation relations of the imaginary quaternions. It is easy to show that submanifolds which are holomorphic Lagrangian with respect to one complex structure are special Lagrangian with respect to another. Hence we can study special Lagrangian fibrations without leaving the realm of algebraic geometry. If M is compact and hyperkähler, its Hodge numbers satisfy $h^{p,q}(M) = h^{n-p,q}(M)$, where n is the complex dimension of M . So mirror symmetry leads us to expect that $h^{p,q}(M) = h^{p,q}(\hat{M})$. We will see that this is indeed the case. It even remains true for the non-compact hyperkähler examples we shall encounter.

The metric inducing this hyperkähler structure is not only hyperkähler, it is flat! So the Riemannian geometry of the situation is not very interesting. What makes things non-trivial is the presence of orbifold singularities. In a few of our cases, these can be resolved crepantly [9], leading to Beauville's examples [5] of compact hyperkähler manifolds, which do carry non-flat metrics. But in most cases, there is no crepant resolution. What we actually evaluate, therefore, are not actual Hodge numbers, but rather orbifold Hodge numbers in the sense of Vafa [50], Zaslow [51], and Batyrev-Dais [4].

It is worth emphasizing that, although many aspects of mirror symmetry remain highly speculative, the results on orbifold Hodge numbers in this paper are mathematically rigorous. Their motivation and interpretation are, of course, more open-ended. To clarify this distinction, the paper has been divided into two parts. The first, comprising sections 1 to 3, consists of background and motivation. The second, comprising sections 4 to 7, contains precise mathematical statements and proofs.

Section 1 gives the necessary background on hyperkähler manifolds and special Lagrangian tori. Sections 2 and 3 introduce the two main classes of examples we shall study. The first is a component of the space of conjugacy classes of commuting pairs of elements in a complex reductive group G . It can be identified, thanks to the work of Hitchin, with the space of Higgs bundles on an elliptic curve. The second, likewise, is a component of the space of conjugacy classes of commuting *quadruples* of elements of a *compact* Lie group K . In each case, the mirror is the same kind of space, only with the group replaced by its Langlands dual.

The next three sections consist of a rigorous formulation and proof of the equality of Hodge numbers predicted by mirror symmetry for these spaces. One first has to normalize the spaces, which is carried out in section 4. Then, since they are not compact or smooth, one needs to define the Hodge numbers judiciously: the suitable definitions, of the so-called orbifold E -polynomials, are recalled in section 5. Finally, the main theorem, showing that the orbifold E -polynomials of the putative mirror partners agree, is proved in section 6.

Section 7 consists of a more or less explicit evaluation of the orbifold E -polynomials for

a group of the form $\mathrm{SL}(n)/\mathbb{Z}_m$. Section 8 contains some concluding remarks and suggestions for further research.

Notation. A few conventions should be mentioned. First, all varieties are over the complex numbers. Second, \otimes means $\otimes_{\mathbb{Z}}$ unless otherwise stated. Finally, the expression “Killing form” is used loosely to mean any nondegenerate symmetric bilinear form on a complex reductive Lie algebra \mathfrak{g} which is compatible with the splitting $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$ and restricts to the Killing form, in the usual sense, on the second factor.

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1 Special Lagrangian tori on hyperkähler manifolds

Let M be a Calabi-Yau manifold of real dimension $2n$, with Kähler form ω and holomorphic n -form $\Omega = \Omega_1 + i\Omega_2$. A real submanifold L of M is said to be *Lagrangian* if $\dim L = n$ and $\omega|_L = 0$, and *special Lagrangian* if $\Omega_2|_L = 0$ as well.

To describe the mirror \hat{M} , Strominger-Yau-Zaslow ask for a real n -manifold N and a map $M \rightarrow N$ whose fiber over a general point $x \in N$ is a special Lagrangian n -torus L_x . The mirror should be the total space of the family over N whose fiber over x is the dual torus $\hat{L}_x = \mathrm{Hom}(\pi_1(L_x), \mathrm{U}(1))$. At least when the map $M \rightarrow N$ has a section, the mirror of \hat{M} can be identified with the original M by double duality. It is not clear, though, how to dualize the singular fibers of the map, or how to define a complex structure on \hat{M} .

Now suppose that M is a *hyperkähler* manifold. This means that it has a metric which is Kähler with respect to three complex structures $J_1, J_2, J_3 : TM \rightarrow TM$ satisfying the commutation relations of the imaginary quaternions. Let $\omega_1, \omega_2, \omega_3$ be the three Kähler forms. Then it is not hard to show that $\omega_2 + i\omega_3$ defines a complex symplectic form on M , which is holomorphic when M is equipped with the complex structure J_1 . In particular, M must have real dimension $4m$ for some m , and $\Omega = (\omega_2 + i\omega_3)^m$ is a nowhere vanishing section of K_M . Therefore M may be regarded as a (possibly non-compact) Calabi-Yau manifold. Also, since M is complex symplectic, $TM \cong T^*M$ holomorphically, so $\Omega^p M \cong \Omega^{2m-p} M$, and hence in the compact case $h^{p,q}(M) = h^{2m-p,q}(M)$.

We will occasionally refer to the identity $M \rightarrow M$, regarded as a non-holomorphic map between different complex structures, as *hyperkähler rotation*.

Let L be a real submanifold of M which is complex Lagrangian with respect to $\omega_2 + i\omega_3$. Then L is actually a complex submanifold with respect to J_1 [28]. Furthermore, since $\omega_2|_L = \omega_3|_L = 0$, L is real Lagrangian with respect to ω_3 , and $(\omega_1 + i\omega_2)^m$ restricts to L as ω_1^m , so that its imaginary part vanishes. Hence L is *special Lagrangian with respect to J_3* .

This fact will be of key importance for us. It means that, if we seek special Lagrangian torus fibrations on a hyperkähler manifold, we need look no further than *holomorphic* Lagrangian torus fibrations in a different complex structure. In particular, the hard analysis usually appearing in the search for special Lagrangian submanifolds can be completely avoided: we can work in a purely algebraic setting.

Even better, there is an obvious source of families of holomorphic Lagrangian torus fibrations on hyperkähler manifolds. Liouville's theorem from Hamiltonian mechanics assures us that, if we have m Poisson-commuting holomorphic functions on M whose derivatives are generically independent, then the map $M \rightarrow \mathbb{C}^m$ they define will have exactly the desired property, provided at least that it is proper. If so, we have what is called an algebraically completely integrable Hamiltonian system.

There is a celebrated system of this kind, the so-called *Hitchin system*. It lives on the moduli space of Higgs bundles on an algebraic curve. In this paper we shall assume that the curve is elliptic. However, curves of genus > 1 will also furnish many interesting examples of mirror pairs. This is the subject of a forthcoming paper of T. Hausel and the author [25].

The spaces we shall consider here will actually be orbifolds, not manifolds. Indeed, their singularities will be exactly what makes the computation of the Hodge numbers non-trivial. However, they will be global quotients of hyperkähler manifolds — indeed, of extremely simple ones.

The torus fibrations coming from algebraically integrable systems are such convenient sources of special Lagrangian tori that it is a little mysterious why they have never been studied before in connection with Strominger-Yau-Zaslow. One reason might be the work of Verbitsky [50] on mirror symmetry and hyperkähler manifolds. Verbitsky showed that a compact hyperkähler manifold is mirror to itself in quite a strong sense: not only are the Hodge numbers self-mirror, which is trivial, but the Yukawa coupling also corresponds to the quantum product (which equals the ordinary cup product in the hyperkähler case), and even the local variations of these structures correspond. This seems to make the search for a mirror into a triviality. However, our examples show, without contradicting Verbitsky's results, that the Strominger-Yau-Zaslow mirror of a compact hyperkähler orbifold, such as the fiber of the sum map $\text{Sym}^m A \rightarrow A$ for an abelian surface A , can be a different space: see Proposition (4.5) below.

2 Higgs bundles on an elliptic curve

Here, then, is the situation we want to consider. Let G be a connected complex reductive algebraic group, let $H \subset G$ be a maximal torus, let $W = N(H)/H$ be the Weyl group, and let $\Lambda \subset \mathfrak{h}$ be the coweight lattice. Also let $\hat{H} = \text{Hom}(H, \mathbb{C}^\times)$ and $\hat{\Lambda} = \text{Hom}(\Lambda, \mathbb{Z})$. There is in some sense canonically associated to G a reductive group \hat{G} , the *Langlands dual*, with maximal torus \hat{H} and coweight lattice $\hat{\Lambda}$.

Let C be an elliptic curve. We will examine the moduli spaces studied by Simpson: $M_{\text{Dol}}(G)$, the moduli space of topologically trivial semistable Higgs G -bundles on C ; $M_{\text{DR}}(G)$, the moduli space of topologically trivial local systems on C with structure group G ; and $M_{\text{B}}(G)$, the identity component of the moduli space of homomorphisms $\pi_1(C) \rightarrow G$ modulo conjugacy. The subscripts stand for Dolbeault, de Rham, and Betti respectively, and are Simpson's notation.

We refer to Simpson [43, 44] for the precise definitions of these spaces, including the correct notions of stability and equivalence. Suffice it to say that a *Higgs G -bundle* is a pair (E, ϕ) , where E is a holomorphic principal G -bundle, and $\phi \in H^0(C, \text{ad } E \otimes K_C)$. This is particularly simple in the present case, since K_C is trivial. A *local system* is a holomorphic

principal G -bundle with an integrable holomorphic connection. A local system is determined up to isomorphism by its holonomy; this induces an isomorphism $M_{\text{DR}}(G) \cong M_{\text{B}}(G)$, but only analytically, since the universal cover of C must be used to construct the isomorphism.

Note also that since $\pi_1(C) = \mathbb{Z} \oplus \mathbb{Z}$, $M_{\text{B}}(G)$ is simply the identity component of the space of commuting pairs of elements of G , modulo conjugacy.

Simpson [43] showed that there is a natural homeomorphism between $M_{\text{Dol}}(G)$ and $M_{\text{DR}}(G)$. It is not a biholomorphism on the smooth locus; rather, Hitchin [26] showed that the two complex structures J_1, J_2 coming from $M_{\text{Dol}}(G)$ and $M_{\text{DR}}(G)$ form part of a hyperkähler structure.

There is a dominant proper morphism, the *Hitchin map*, which takes $M_{\text{DR}}(G)$ to a vector space, and is defined by evaluating a set of generators for the invariant polynomials of \mathfrak{g} on ϕ . The map in question therefore takes values in \mathfrak{g}/G , which according to a classic theorem of Chevalley [49, 4.9] is a vector space and is isomorphic to \mathfrak{h}/W .

The fibers of the Hitchin map have been much studied and are known to be generically abelian varieties [27]. In our case, the fiber over a regular element of \mathfrak{h}/W can be identified as $C \otimes \Lambda$. This can, for example, be extracted from the work of Donagi-Gaitsgory [15] or Faltings [16]: according to their recipe one gets $H^1(C, \mathcal{O}(H))$, where $\mathcal{O}(H)$ denotes the sheaf of germs of holomorphic maps to H . Since $H = \mathbb{C}^\times \otimes \Lambda$, this is naturally isomorphic to $H^1(C, \mathcal{O}^\times) \otimes \Lambda$; but since we are dealing only with topologically trivial bundles, we must take only the identity component, which is $C \otimes \Lambda$.

How are the moduli spaces for G and \hat{G} related? Exactly as one would hope for the Strominger-Yau-Zaslow construction to apply! Since W acts orthogonally with respect to the Killing form, $\mathfrak{h} \cong \hat{\mathfrak{h}}$ as representations of W . So $\mathfrak{h}/W \cong \hat{\mathfrak{h}}/W$: the Hitchin maps have isomorphic targets. On the other hand, $C \otimes \Lambda = \text{Pic}^0(C \otimes \hat{\Lambda})$: the generic fibers of the two Hitchin maps are dual. Furthermore, the Higgs bundles whose underlying bundles are trivial furnish a section of each Hitchin map.

Hence $M_{\text{DR}}(G)$ and $M_{\text{DR}}(\hat{G})$ satisfy the requirements to be Strominger-Yau-Zaslow mirror partners. We might conjecture, therefore, that their Hodge numbers are equal. But what is meant by the Hodge numbers of varieties which are not compact or smooth? The right answer seems to be the *stringy E-polynomial* of Batyrev-Dais [4]. Our chief goal, attained in §6, will be to show that our moduli spaces have the same stringy E -polynomials as their putative mirror partners.

One technicality stands in the way, however. The minimum requirements for varieties to have well-defined stringy E -polynomials are not altogether clear, but they certainly must be *normal*. It is by no means obvious that the moduli spaces discussed above meet this minimum requirement.

The solution is to replace every space in the story by its normalization. This motivates §4, in which it is shown that the normalizations of $M_{\text{Dol}}(G)$, $M_{\text{DR}}(G)$, and $M_{\text{B}}(G)$ are $(T^*C \otimes \Lambda)/W$, $(H \times H)/W$, and $(H \times H)/W$ respectively. The cotangent bundle $T^*C = C \times \mathbb{C}$ has an obvious hyperkähler structure, which induces one on $T^*C \otimes \Lambda$, so $(T^*C \otimes \Lambda)/W = ((C \otimes \Lambda) \times \mathfrak{h})/W$ is a hyperkähler orbifold. The second complex structure on T^*C is $\mathbb{C}^\times \times \mathbb{C}^\times$, so the second complex structure on $(T^*C \otimes \Lambda)/W$ is $((\mathbb{C}^\times \times \mathbb{C}^\times) \otimes \Lambda)/W = (H \times H)/W$. The Hitchin map lifts to the projection $((C \otimes \Lambda) \times \mathfrak{h})/W \rightarrow \mathfrak{h}/W$, whose generic fiber is clearly $C \otimes \Lambda$. The mirror transformation can therefore be summarized as

follows. The vertical arrows denote hyperkähler rotation, and the diagonal arrows are the normalized Hitchin maps.

$$\begin{array}{ccc}
\frac{H \times H}{W} & & \frac{\hat{H} \times \hat{H}}{W} \\
\downarrow & & \downarrow \\
\frac{T^*C \otimes \Lambda}{W} & & \frac{T^*C \otimes \hat{\Lambda}}{W} \\
\searrow & & \swarrow \\
& \frac{\mathfrak{h}}{W} &
\end{array}$$

We hope, and indeed will show in §6, that the stringy E -polynomials of the spaces in the top row agree.

3 Principal bundles on an abelian surface

Discussions with Jim Bryan in October 1998 revealed that there is an analogue of the above situation where all the spaces involved are compact. They are, in fact, precisely the spaces studied in the recent work of Bryan-Donagi-Leung [9].

Let $A = \mathrm{U}(1)^4$ be a 4-torus with a hyperkähler structure compatible with the flat metric. This is easily found: just choose any linear isomorphism from $\mathfrak{u}(1)^4$ to the quaternions. For convenience, we will suppose that in one complex structure, A splits as a product $C \times D$ of two elliptic curves. However, many of our considerations would extend to the more general case of an abelian surface which is an extension of one elliptic curve by another.

Consider the moduli space of topologically trivial semistable principal G -bundles on A . Let $M(G)$ be the connected component containing the trivial bundle. The correspondence theorem relating holomorphic bundles to Hermitian-Einstein connections [14, 47] has been generalized to arbitrary compact structure groups by Ramanathan-Subramanian [37]. So if K is a compact Lie group whose complexification is G , then every equivalence class of topologically trivial semistable G -bundles has a representative carrying a flat K -connection, unique up to conjugacy.

Hence $M(G)$ is homeomorphic to a connected component of the space of homomorphisms $\pi_1(A) \rightarrow K$ modulo conjugacy, that is, the space of commuting *quadruples* of elements of K , modulo conjugacy. As shown by Borel-Friedman-Morgan [8, 2.3.2], the connected component of the identity consists precisely of those quadruples which all belong to a common maximal torus.

Since A has several complex structures, this endows the moduli space $M(G)$ with several complex structures. One would expect that they give rise to a hyperkähler structure on the smooth locus, but this result does not seem to appear in the literature except for $G = \mathrm{GL}(n, \mathbb{C})$ [34]. However, this lacuna will not be serious as the normalization of $M(G)$ will manifestly be a hyperkähler orbifold. In fact, it is shown in §4 that it is exactly the quotient

$(\hat{A} \otimes \Lambda)/W$. In the present case this can be identified with $(A \otimes \Lambda)/W$, since $C \times D$ is self-dual and hence so is A in the other Kähler structures.

When A has the complex structure of $C \times D$, then restriction to C times a point defines a morphism from $M(G)$ to the moduli space of semistable G -bundles on C . (That semistability is preserved follows from the correspondence theorem.) The fibers of this morphism are not as well-studied as those of the Hitchin map. But the situation is clarified by passing to the normalization. The space of topologically trivial semistable G -bundles on an elliptic curve C is shown by Friedman-Morgan [17] and Laszlo [31] to be $(C \otimes \Lambda)/W$. (When G is simply connected, this is the space shown by Looijenga [32] and Bernshtein-Shvartsman [6] to be a weighted projective space.) The normalization of the restriction map is easily seen to be just the map $(A \otimes \Lambda)/W \rightarrow (C \otimes \Lambda)/W$ induced by the projection $A \rightarrow C$.

The generic fiber is clearly the abelian variety $D \otimes \Lambda$. Moreover, there is an obvious candidate for the dual fibration, namely the quotient of $(D \otimes \hat{\Lambda}) \times (C \otimes \Lambda)$ by W . Since this torus has tangent space naturally isomorphic to that of $A \otimes \Lambda$, it too carries a natural hyperkähler structure. Denote \mathcal{B} the space obtained by hyperkähler rotation. It does not split naturally, but it nevertheless carries an action of W , and \mathcal{B}/W is the mirror of $(A \otimes \Lambda)/W$.

The story so far is summarized by the left-hand side of the diagram below, with the vertical arrows denoting hyperkähler rotation as before. But the roles of Λ and $\hat{\Lambda}$ are completely interchangeable, so the diagram has a right-hand side as well. For that matter, the diagram could continue with $\hat{\mathcal{B}}/W$, to make a closed chain with four links.

$$\begin{array}{ccccc}
\frac{A \otimes \Lambda}{W} & & \frac{\mathcal{B}}{W} & & \frac{A \otimes \hat{\Lambda}}{W} \\
\downarrow & & \downarrow & & \downarrow \\
\frac{(C \times D) \otimes \Lambda}{W} & & \frac{(D \otimes \hat{\Lambda}) \times (C \otimes \Lambda)}{W} & & \frac{(C \times D) \otimes \hat{\Lambda}}{W} \\
& \searrow & \swarrow & \searrow & \swarrow \\
& \frac{C \otimes \Lambda}{W} & & \frac{D \otimes \hat{\Lambda}}{W} &
\end{array}$$

Thus each space in the top row has *two* special Lagrangian torus fibrations, and hence two mirrors! Under the circumstances, it is tempting to guess that perhaps these two mirrors are isomorphic. However, this is generally not the case. Indeed, we shall see in Proposition (4.5) that for $G = \mathrm{SL}(3)$, $(A \otimes \Lambda)/W$ has a crepant resolution while $(A \otimes \hat{\Lambda})/W$ does not.

Once again, we will be able to verify the prediction of mirror symmetry that the stringy E -polynomials of these spaces will be the same.

4 The normalizations of the moduli spaces

As in the previous two sections, let G be a complex connected reductive group, Λ its coweight lattice, W the Weyl group, and let C be an elliptic curve. We will describe the normalizations of Simpson's moduli spaces over C as follows.

(4.1) Theorem. *The normalizations of $M_{\text{Dol}}(G)$, $M_{\text{DR}}(G)$, and $M_{\text{B}}(G)$ are naturally isomorphic to $(T^*C \otimes \Lambda)/W$, $(J \otimes \Lambda)/W$, and $(H \times H)/W$ respectively. Here J denotes the unique algebraic group which is a non-trivial extension of C by the affine line.*

To construct the morphisms that will turn out to be the normalizations, consider first the case $G = \mathbb{C}^\times$. Clearly $M_{\text{Dol}}(\mathbb{C}^\times) = T^*C = C \times \mathbb{C}$ and $M_{\text{B}}(\mathbb{C}^\times) = \mathbb{C}^\times \times \mathbb{C}^\times$. Moreover, there is an algebraic group epimorphism from $M_{\text{DR}}(\mathbb{C}^\times)$ to the moduli space of topologically trivial holomorphic \mathbb{C}^\times -bundles on C , which of course is isomorphic to C . The kernel consists of the holomorphic connections on the trivial bundle, which can be identified with $H^1(C, \mathcal{O}) \cong \mathbb{C}$. The extension is not split, since $M_{\text{DR}}(\mathbb{C}^\times)$ is analytically isomorphic to $M_{\text{B}}(\mathbb{C}^\times) = \mathbb{C}^\times \times \mathbb{C}^\times$ which cannot contain a complete curve. But there is well known to be only a single non-trivial extension J of C by \mathbb{C} : see for example Serre [42, VII §3].

Now for general connected reductive G , let $H = \mathbb{C}^\times \otimes \Lambda$ be a Cartan subgroup. Then $M_{\text{Dol}}(H) = T^*C \otimes \Lambda$, $M_{\text{DR}}(H) = J \otimes \Lambda$, and $M_{\text{B}}(H) = (\mathbb{C}^\times \times \mathbb{C}^\times) \otimes \Lambda = H \times H$. Extension of the structure group induces a morphism $M_{\text{Dol}}(H) \rightarrow M_{\text{Dol}}(G)$, and likewise for the de Rham and Betti spaces. The Weyl group acts by outer automorphisms on H that extend to inner automorphisms on G , so it acts on $M_{\text{Dol}}(H)$ via its action on Λ , but trivially on $M_{\text{Dol}}(G)$. The aforementioned morphism therefore descends to a morphism $\rho_{\text{Dol}} : (T^*C \otimes \Lambda)/W \rightarrow M_{\text{Dol}}(G)$. Similarly, there are morphisms ρ_{DR} and ρ_{B} .

Note that there is a commutative diagram of continuous maps

$$\begin{array}{ccccc} \frac{T^*C \otimes \Lambda}{W} & \longleftrightarrow & \frac{J \otimes \Lambda}{W} & \longleftrightarrow & \frac{H \times H}{W} \\ \downarrow \rho_{\text{Dol}} & & \downarrow \rho_{\text{DR}} & & \downarrow \rho_{\text{B}} \\ M_{\text{Dol}}(G) & \longleftrightarrow & M_{\text{DR}}(G) & \longleftrightarrow & M_{\text{B}}(G). \end{array}$$

(4.2) Proposition. *The morphisms ρ_{Dol} , ρ_{DR} and ρ_{B} are finite bijections.*

Proof. Being finite is equivalent to being proper with finite fibers. Everything is therefore a topological statement. Since the rows in the diagram above consist of homeomorphisms, it suffices to prove the proposition for one of the three moduli spaces. We select the Betti space.

Both $(H \times H)/W$ and $M_{\text{B}}(G)$ have natural morphisms to $H/W \times H/W$. The projection $f : (H \times H)/W \rightarrow H/W \times H/W$ is the obvious one, and the morphism $g : \mu^{-1}(I)/G \rightarrow H/W \times H/W$ is equally obvious once H/W is identified [45, 6.4] with the quotient G/G , where G acts on itself by conjugation. These morphisms satisfy $f = g \circ \rho_{\text{B}}$. And f is finite: the coordinate ring of $H \times H$ is finitely generated over that of $H/W \times H/W$, so the same

is true of its submodule, the coordinate ring of $(H \times H)/W$. Hence ρ is a proper morphism [24, II 4.8(e)] of affine varieties, and hence finite [24, II Ex. 4.6].

The work of Richardson [39] implies that if G is reductive, connected and simply connected, then the set of commuting pairs in $G \times G$ is irreducible, and hence $M_B(G)$ is irreducible. It follows that $M_B(G)$ is still irreducible even if G is not simply connected. For it was defined to be the connected component of the space of commuting pairs containing (I, I) . This is surjected on by $Z_0(G)^2 \times M_B(\tilde{G})$, where $Z_0(G)$ is the identity component of the center, and \tilde{G} is the universal cover of the commutator subgroup.

Hence to show that the proper map ρ_B is surjective, it suffices to show that it is dominant. Let h_1, h_2 be any commuting regular semisimple elements of a Cartan subgroup $H \subset G$. The kernels of ad_{h_1} and ad_{h_2} on \mathfrak{g} are both precisely \mathfrak{h} . Hence there exists a neighborhood of $(h_1, h_2) \in M_B(G)$, in the complex topology, such that for (g_1, g_2) in this neighborhood, $\ker \text{ad}_{g_1}$ and $\ker \text{ad}_{g_2}$ coincide and have dimension equal to the rank of G . We may also assume that g_1 and g_2 remain regular semisimple elements in this neighborhood, so their centralizers remain tori. The Lie algebras of these centralizers are both $\ker \text{ad}_{g_1}$, so they are both the same maximal torus and hence g_1 and g_2 belong to the same Cartan subgroup. Hence (g_1, g_2) is in the image of ρ_B . Therefore ρ_B surjects onto a neighborhood in the complex topology, and hence is dominant.

To show ρ_B is injective, we follow an argument of Borel [7]. Suppose that $\rho_B(h_1, h_2) = \rho_B(h'_1, h'_2)$. This means that $(h_1, h_2) \in H \times H$ is conjugate to $(h'_1, h'_2) \in H \times H$ by some $g \in G$. The centralizer of h_1 and h_2 , say $Z(h_1, h_2) \subset G$, is a reductive subgroup. This follows, for example, from 26.2A of Humphreys [29], since the proof there is valid not only for a subtorus, but for any subset. Then H and gHg^{-1} are maximal tori in $Z(h_1, h_2)$, so gHg^{-1} is conjugate to H by some $g' \in Z(h_1, h_2)$. Then $g'g \in N(H)$ conjugates (h_1, h_2) to (h'_1, h'_2) , so they represent the same point in $(H \times H)/W$. \square

Proof of Theorem (4.1). First, note that, as varieties surjected on by irreducible varieties, all three moduli spaces are irreducible.

For any x in the dense set of the proposition, the fiber $\rho_B^{-1}(x)$ has a single closed point. Since ρ_B is finite, the locus where its fibers are reduced is open, and, of course, it contains the generic point. Hence over a nonempty open set, the fiber of ρ is a single reduced point. Therefore ρ_B is birational. As the quotient of a smooth variety by a finite group, $(H \times H)/W$ is certainly normal, so ρ_B lifts to a birational finite morphism $\tilde{\rho}_B : (H \times H)/W \rightarrow \tilde{M}_B(G)$ on the normalizations. This is an isomorphism by Zariski's Main Theorem.

The same argument applies to the Dolbeault and de Rham spaces. \square

Exactly the same methods can be used to prove a similar theorem on the space $M(G)$ of G -bundles on an abelian surface A which was discussed in §3.

(4.3) Theorem. *The normalization of $M(G)$ is naturally isomorphic to $(\hat{A} \otimes \Lambda)/W$.*

The proof is parallel to that of Theorem (4.1). One defines a morphism $\rho : (\hat{A} \otimes \Lambda)/W \rightarrow M(G)$, and it then suffices to prove the following analogue of (4.2). Finiteness follows automatically since the domain is compact.

(4.4) Proposition. *The morphism ρ is a bijection.*

Proof. This can be deduced conveniently using the correspondence with flat connections. So let K be a compact Lie group whose complexification is G , $T \subset K$ the maximal torus whose complexification is the Cartan subgroup H . Then $M(G)$ gets identified with the identity component of the space of commuting quadruples in K modulo conjugacy, and $M(H) = \hat{A} \otimes \Lambda$ gets identified with T^4 .

Surjectivity is a consequence of the aforementioned result of Borel-Friedman-Morgan [8, 2.3.2]. For injectivity, we again follow the argument of Borel. If two quadruples of elements of T have the same image in $M(G)$, then there exists $g \in K$ conjugating one quadruple to another. Let $Z \subset K$ be the centralizer of the first quadruple. This is compact and contains T and gTg^{-1} as maximal tori, so there exists $g' \in T$ conjugating gTg^{-1} to T . Then $g'g \in N(T)$ and conjugates one quadruple to another, so the two quadruples represent the same point in T^4/W . \square

To show that our mirror transformation is non-trivial, here is an example of a space that certainly differs from its mirror.

(4.5) Proposition. *The normalized moduli spaces of Theorems (4.1) and (4.3) have crepant resolutions for $G = \mathrm{SL}(3)$, but not for $G = \mathrm{PGL}(3)$.*

Proof. Let A be any 2-dimensional connected abelian algebraic group, and let $G = \mathrm{SL}(3)$. The Weyl group is the symmetric group S_3 , and the torus $A \otimes \Lambda$ can be regarded as the kernel of the sum map $A^3 \rightarrow A$. The quotient $(A \otimes \Lambda)/S_3$ is therefore the fiber of the sum map $\mathrm{Sym}^3 A \rightarrow A$, where $\mathrm{Sym}^3 A$ is the symmetric product A^3/S_3 . As explained by Beauville [5], this has a crepant resolution, namely the fiber of the corresponding map $\mathrm{Hilb}^3 A \rightarrow A$, where $\mathrm{Hilb}^3 A$ is the Hilbert scheme parametrizing subschemes of dimension 0 and length 3.

The torus $A \otimes \hat{\Lambda}$ is, of course, isomorphic to $A \otimes \Lambda$. However, the action of S_3 on $A \otimes \hat{\Lambda}$ is different. In terms of a basis for $\hat{\Lambda}$, which splits $A \otimes \hat{\Lambda}$ as $A \times A$, the S_3 -action is generated by the elements $(x, y) \mapsto (-x - y, x)$ of order 3 and $(x, y) \mapsto (-y, -x)$ of order 2. So for any nonzero $x \in A$ with $3x = 0$, $(x, x) \in A^2$ has stabilizer \mathbb{Z}_3 . Therefore the singularity of $(A \otimes \hat{\Lambda})/S_3$ at this point is analytically isomorphic to the quotient $\mathbb{C}^4/\mathbb{Z}_3$, where $\xi = e^{2\pi i/3}$ acts by $(w, x, y, z) \mapsto (\xi w, \xi^2 x, \xi y, \xi^2 z)$. This has no crepant resolution [38, 5.4]. Hence neither does $(A \otimes \hat{\Lambda})/S_3$: the smoothness and discrepancies of resolutions are analytical invariants, since they can be computed on the completed local rings. \square

5 Review of orbifold E -polynomials

Let W be a finite group acting on a smooth projective variety X . Then for each $p, q \geq 0$, W acts on $H^{p,q}(X)$. Denote the character of this representation by $h_W^{p,q}(X) : W \rightarrow \mathbb{C}$. One should think of the character as a sort of equivariant Betti number. Its average value

$$\bar{h}_W^{p,q}(X) = \frac{1}{|W|} \sum_{w \in W} h_W^{p,q}(X)(w)$$

is the dimension of the W -invariant part of $H^{p,q}(X)$.

For any $w \in W$, denote X^w the fixed-point set of w ; this is a smooth subvariety. Suppose that the action of each $w \in W$ on $K_X|_{X^w}$ is trivial. Then Vafa [48] and Zaslow [51] define *orbifold Hodge numbers* associated to the quotient orbifold X/W as follows. For any $x \in X^w$, w acts on $T_x X$ with weights

$$e^{2\pi i \alpha_1}, \dots, e^{2\pi i \alpha_n},$$

where $\alpha_1, \dots, \alpha_n \in [0, 1)$. Define the *fermionic shift* $F(w)$ to be $\alpha_1 + \dots + \alpha_n$. This is a locally constant function on X^w , and it is integer-valued due to the assumption on $K_X|_{X^w}$. To simplify the notation, assume that it is constant, as will be true in the examples we study. Then define the *orbifold Hodge numbers* as

$$h_{\text{orb}}^{p,q}(X/W) = \sum_{\{w\}} \bar{h}_{C(w)}^{p-F(w), q-F(w)}(X^w),$$

where $C(w)$ denotes the centralizer of w , and the sum runs over the conjugacy classes of W .

Since some of our moduli spaces are non-compact, we wish to generalize these notions to the case where X is a smooth *quasi-projective* variety. The cohomology of X then no longer carries a pure Hodge structure, but Deligne [12, 13] constructs a canonical *mixed Hodge structure* on the compactly supported cohomology, $H_{\text{cpt}}^k(X)$. That is, there are two canonical filtrations on $H_{\text{cpt}}^k(X)$, so that we may write the Betti number as a sum $h_{\text{cpt}}^k = \sum_{p,q} h^{p,q;k}$, where $h^{p,q;k}$ are the dimensions of the quotients $H^{p,q;k}$ associated to these filtrations [12, 2.3.7]. In the smooth projective case, $h^{p,q;k} = 0$ unless $p+q = k$. If we define

$$e^{p,q} = \sum_{k \geq 0} (-1)^k h^{p,q;k},$$

then the so-called *E-polynomial*

$$E(X) = \sum_{p,q} e^{p,q}(X) u^p v^q$$

enjoys some remarkable properties. Specifically, it is additive for disjoint unions of locally closed subvarieties, and multiplicative for Zariski locally trivial fibrations.

The action of a finite group W on X preserves the mixed Hodge structure, since it is canonical. Hence the spaces $H^{p,q;k}(X)$ are representations of W . As before, let $h_W^{p,q;k}(X)$ be their characters, and let $\bar{h}_W^{p,q;k}(X)$ be the average values of these characters. Let $e_W^{p,q}$ and $E_W(X)$ be the W -equivariant versions of the expressions above, and let $\bar{E}_W(X)$ be the same as $E_W(X)$, but with h replaced by \bar{h} . Note that $E_W(X \times Y) = E_W(X)E_W(Y)$, but that the corresponding statement is usually false for the average values \bar{E}_W .

The *orbifold E-polynomial* can now be defined as

$$E_{\text{orb}}(X/W) = \sum_{\{w\}} \bar{E}_{C(w)}(X^w) (uv)^{F(w)}.$$

In the projective case, this reduces to

$$E_{\text{orb}}(X/W) = \sum_{p,q} h_{\text{orb}}^{p,q}(X/W)(-u)^p(-v)^q.$$

The following theorem is proved by Batyrev-Dais [4, 6.14].

(5.1) Theorem. *The orbifold E -polynomial depends only on the orbifold structure of X/W .*

In fact, they show that it agrees with the more generally defined notion of the “stringy E -polynomial”. Their proof as stated assumes that X is projective (or rather compact Kähler), but it applies without change to the quasi-projective case.

6 Equality of the orbifold E -polynomials

We are ready to state and prove our main result, and then to show how it implies the equalities predicted by mirror symmetry.

(6.1) Theorem. *Let A be a connected abelian algebraic group, let W be a finite group acting orthogonally on a lattice Λ equipped with a positive definite rational quadratic form, and let B be a smooth variety also acted on by W . If $X = (A \otimes \Lambda) \times B$ and $\hat{X} = (A \otimes \hat{\Lambda}) \times B$ have the natural actions of W , then $E_{\text{orb}}(X/W) = E_{\text{orb}}(\hat{X}/W)$.*

Even in the case $B = 0$, the two quotients X/W and \hat{X}/W will generally not be the same: see Proposition (4.5). Moreover, the theorem is certainly false if $\hat{\Lambda}$ is replaced by some other lattice isogenous to Λ : see for example Theorem (7.1).

To prove the theorem, we will show that the sums defining the two orbifold E -polynomials agree term by term: that is, for each $w \in W$, the contributions of X^w and \hat{X}^w to the sums will be identical. To begin with, consider the fermionic shifts.

(6.2) Lemma. *The fermionic shifts $F(w)$ and $\hat{F}(w)$ for the action of $w \in W$ on $A \otimes \Lambda$ and $A \otimes \hat{\Lambda}$ are constant and equal.*

Proof. Every fixed point $(A \otimes \Lambda)^w$ is a subgroup. Translation by any element of this subgroup commutes with the action of w , so $F(w)$ is constant. The Lie algebra of this subgroup is $(\mathfrak{a} \otimes \Lambda)^w \subset (\mathfrak{a} \otimes \Lambda)$. Since \mathfrak{a} is a complex vector space, this equals $(\mathfrak{a} \otimes_{\mathbb{C}} \mathfrak{h})^w$ where $\mathfrak{h} = \mathbb{C} \otimes \Lambda$ is the Cartan subalgebra of \mathfrak{g} . The Killing form induces an isomorphism $\mathfrak{h} \cong \hat{\mathfrak{h}}$; since W preserves the Killing form, it acts identically on these two spaces. \square

(6.3) Lemma. *The identity components of $(A \otimes \Lambda)^w$ and $(A \otimes \hat{\Lambda})^w$ have the same $E_{C(w)}$ -polynomials.*

Proof. Since the action of $w-1$ on Λ can be row-reduced over $\Lambda \otimes \mathbb{Q}$, a basis for $\ker(w-1) = \mathfrak{h}^w \subset \mathfrak{h} = \mathbb{C} \otimes \Lambda$ can be found in Λ itself. Hence $\mathfrak{h}^w = \mathbb{C} \otimes \Lambda^w$. Consequently $(\mathfrak{a} \otimes \Lambda)^w =$

$\mathfrak{a} \otimes \Lambda^w$. The identity component of $(A \otimes \Lambda)^w$ is the connected subgroup of $A \otimes \Lambda$ with Lie algebra $(\mathfrak{a} \otimes \Lambda)^w$. It therefore is nothing but $A \otimes \Lambda^w$.

Both Λ and $\hat{\Lambda}$ can be regarded as subgroups of $\mathfrak{t} = \mathbb{R} \otimes \Lambda$. Since Λ^w and $\hat{\Lambda}^w$ then have finite index in the subgroup of \mathfrak{t} they jointly generate, both $A \otimes \Lambda^w$ and $A \otimes \hat{\Lambda}^w$ are quotients of the same abelian group by a finite, $C(w)$ -invariant subgroup. This induces an isomorphism of their compactly supported cohomology preserving the $C(w)$ -action and the mixed Hodge structure. \square

Denote the identity component of $(A \otimes \Lambda)^w$ by $(A \otimes \Lambda)_0^w$. The complex cohomology of $(A \otimes \Lambda)^w$, as a representation of $C(w)$ carrying a mixed Hodge structure, satisfies

$$H^*((A \otimes \Lambda)^w) \cong H^*((A \otimes \Lambda)_0^w) \otimes \mathbb{C}[\pi_0((A \otimes \Lambda)^w)].$$

Here π_0 denotes the group of components, and $\mathbb{C}[\]$ denotes a group algebra. The equivariant E -polynomials therefore satisfy

$$E_{C(w)}((A \otimes \Lambda)^w) = E_{C(w)}((A \otimes \Lambda)_0^w) \chi(\mathbb{C}[\pi_0((A \otimes \Lambda)^w)]),$$

where χ denotes a character.

Theorem (6.1) follows easily from this, together with the two lemmas above, and the following.

(6.4) Proposition. *As representations of $C(w)$,*

$$\mathbb{C}[\pi_0((A \otimes \Lambda)^w)] \cong \mathbb{C}[\pi_0((A \otimes \hat{\Lambda})^w)].$$

In fact we will see that the groups of components are canonically *dual* to each other: that is, one is the group of characters of the other, and the $C(w)$ -action respects this duality.

To determine the group of components $\pi_0((A \otimes \Lambda)^w)$, it is convenient to work over the real numbers. Write A as a product of real lines and circles: $A = \mathbb{R}^c \times \mathrm{U}(1)^d$, for some integers $c, d \geq 0$. The W -action on $A \otimes \Lambda$ of course respects this decomposition, so

$$(A \otimes \Lambda)^w = ((\mathbb{R} \otimes \Lambda)^w)^c \times ((\mathrm{U}(1) \otimes \Lambda)^w)^d.$$

The first factor is a linear subspace, so contributes nothing to the component group. As for the second factor, note that $\mathrm{U}(1) \otimes \Lambda$ is nothing but a maximal torus T of the compact group K . So as sets with $C(w)$ -action,

$$(6.5) \quad \pi_0((A \otimes \Lambda)^w) = \pi_0((T^w)^d).$$

We will not mention the action of $C(w)$ in what follows, but all of our constructions are sufficiently natural to work $C(w)$ -equivariantly.

Let $\mathfrak{t} = \mathbb{R} \otimes \Lambda$. Then there is an exact sequence of W -modules

$$0 \longrightarrow \Lambda \longrightarrow \mathfrak{t} \longrightarrow T \longrightarrow 0.$$

Moreover, the Killing form provides an inner product on \mathfrak{t} . An element $w \in W$ acts on \mathfrak{t} by a linear transformation L which is orthogonal with respect to the Killing form, so its

action on the dual $\hat{\mathfrak{t}}$, via the adjoint inverse, is the same once we identify $\mathfrak{t} = \hat{\mathfrak{t}}$. On the other hand, there is also an exact sequence

$$0 \longrightarrow \hat{\Lambda} \longrightarrow \hat{\mathfrak{t}} \longrightarrow \hat{T} \longrightarrow 0.$$

Define $F : \mathfrak{t} \rightarrow \mathfrak{t}$ by $F(v) = L(v) - v$. Then $T^w = F^{-1}(\Lambda)/\Lambda$. Similarly $\hat{T}^w = F^{-1}(\hat{\Lambda})/\hat{\Lambda}$. Applying F induces a natural homomorphism $\psi : T^w \rightarrow \Lambda/F(\Lambda)$.

(6.6) Lemma. *The map ψ induces a natural isomorphism $\pi_0(T^w) = \text{Tor}(\Lambda/F(\Lambda))$.*

Proof. It suffices to show that $\ker \psi = T_0^w$ and $\text{im } \psi = \text{Tor}(\Lambda/F(\Lambda))$.

Choose $v \in F^{-1}(\Lambda)$ representing an element of $T^w = F^{-1}(\Lambda)/\Lambda$. If $\psi(v) = 0$, then $F(v) \in F(\Lambda)$, so $v \in \ker F + \Lambda$. Hence the element it represents in $F^{-1}(\Lambda)/\Lambda$ belongs to $\ker F/(\Lambda \cap \ker F)$, which is precisely the identity component of $F^{-1}(\Lambda)/\Lambda$. Therefore $\ker \psi \subset T_0^w$. The other inclusion is easy.

Since ψ factors through the finite group $T^w/\pi_0(T^w)$, its image is certainly contained in the torsion part of $\Lambda/F(\Lambda)$. On the other hand, $\text{Tor}(\Lambda/F(\Lambda)) = (\Lambda \cap F(\mathfrak{t}))/F(\Lambda)$, since $\text{Tor } \Lambda = 0$ and $F(\mathfrak{t})$ is the linear span of $F(\Lambda)$. And if $u \in \Lambda \cap F(\mathfrak{t})$, then certainly there exists $v \in F^{-1}(\Lambda)$ such that $u = F(v)$. Hence ψ surjects onto $\text{Tor}(\Lambda/F(\Lambda))$. \square

(6.7) Lemma. *There is a natural isomorphism*

$$\frac{\Lambda}{F(\Lambda)} = \text{Hom} \left(\frac{F^{-1}(\hat{\Lambda})}{\hat{\Lambda}}, \mathbb{U}(1) \right).$$

Proof. We show first that as subsets of \mathfrak{t} , $F(\Lambda) = \text{Hom}(F^{-1}(\hat{\Lambda}), \mathbb{Z})$. For x is in the right-hand side if and only if $\langle x, y \rangle \in \mathbb{Z}$ whenever $y \in F^{-1}(\hat{\Lambda})$, that is, whenever $\langle Fy, z \rangle \in \mathbb{Z}$ for all $z \in \Lambda$. But

$$\langle Fy, z \rangle = \langle Ly, z \rangle - \langle y, z \rangle = \langle y, L^{-1}z \rangle - \langle y, z \rangle = \langle y, F'z \rangle$$

for $F' = L^{-1} - 1$. So the right-hand side is double-dual to, and hence equals, $F'(\Lambda)$. But $F' = -FL^{-1}$, and L^{-1} preserves Λ , so $F'(\Lambda) = F(\Lambda)$.

Now apply $\text{Hom}(\cdot, \mathbb{Z})$ to

$$0 \longrightarrow \hat{\Lambda} \longrightarrow F^{-1}(\hat{\Lambda}) \longrightarrow \hat{T}^w \longrightarrow 0$$

to find

$$0 \longrightarrow \text{Hom}(\hat{T}^w, \mathbb{Z}) \longrightarrow \text{Hom}(F^{-1}(\hat{\Lambda}), \mathbb{Z}) \longrightarrow \text{Hom}(\hat{\Lambda}, \mathbb{Z}) \longrightarrow \text{Ext}^1(\hat{T}^w, \mathbb{Z}) \longrightarrow 0.$$

Now $\text{Hom}(\hat{T}^w, \mathbb{Z}) = 0$ since \hat{T}^w is a compact abelian group, and the next two terms have been identified as $F(\Lambda)$ and Λ respectively, so there is a natural isomorphism

$$\text{Ext}^1(\hat{T}^w, \mathbb{Z}) = \frac{\Lambda}{F(\Lambda)}.$$

On the other hand, apply $\mathrm{Hom}(\hat{T}^w, \cdot)$ to

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathrm{U}(1) \longrightarrow 0$$

to find

$$0 \longrightarrow \mathrm{Hom}(\hat{T}^w, \mathbb{Z}) \longrightarrow \mathrm{Hom}(\hat{T}^w, \mathbb{R}) \longrightarrow \mathrm{Hom}(\hat{T}^w, \mathrm{U}(1)) \longrightarrow \mathrm{Ext}^1(\hat{T}^w, \mathbb{Z}) \longrightarrow 0.$$

The first two terms are certainly 0 since T^w is a compact abelian group, so the last two terms are naturally isomorphic. Applying $\mathrm{Hom}(\cdot, \mathrm{U}(1))$ to both sides yields the desired result. \square

(6.8) Lemma. *If Γ is any finitely generated abelian group, such as $\Lambda/F(\Lambda)$, then there is a natural isomorphism*

$$\pi_0(\mathrm{Hom}(\Gamma, \mathrm{U}(1))) = \mathrm{Hom}(\mathrm{Tor} \Gamma, \mathrm{U}(1)).$$

Proof. The map is just restriction of homomorphisms to the torsion part; that this is well-defined and induces the desired isomorphism follows easily from the classification of finitely generated abelian groups. \square

Proof of Proposition (6.4). Putting together the three preceding lemmas, we conclude that there is a isomorphism

$$\pi_0(\hat{T}^w) \cong \mathrm{Hom}(\pi_0(T^w), \mathrm{U}(1)),$$

which is natural, and, in particular, compatible with the $C(w)$ -action. Consequently,

$$\mathbb{C}[\pi_0(\hat{T}^w)] \cong \mathbb{C}[\pi_0(T^w)]$$

as representations of $C(w)$: the map is given by the finite Fourier transform $f \mapsto \hat{f}$, namely

$$\hat{f}(h) = \sum_{k \in \pi_0(T^w)} h(k) f(k).$$

Now take d th powers and apply (6.5). This completes the proof of Proposition (6.4), and hence of Theorem (6.1). \square

It is now easy to verify the predictions of mirror symmetry for the normalized spaces of §2. Just take $A = J$, $B = 0$ in Theorem (6.1). This gives $E_{\mathrm{orb}}(\tilde{M}_{\mathrm{DR}}(G)) = E_{\mathrm{orb}}(\tilde{M}_{\mathrm{DR}}(\hat{G}))$, as desired. Similar identities hold for the Dolbeault and Betti spaces.

If instead A is taken to be an elliptic curve D , and B to be $C \otimes \Lambda$ for another elliptic curve C , the theorem shows that the spaces in the second row of the diagram from §3 all have the same E_{orb} . To draw the same conclusion for the spaces in the first row, as predicted by mirror symmetry, we need one last lemma, which shows that the hyperkähler rotation will not change E_{orb} .

(6.9) Lemma. *Suppose the Weyl group W acts on a real torus \mathcal{B} whose Lie algebra \mathfrak{b} is W -equivariantly isomorphic to $\mathbb{H} \otimes \Lambda$, where \mathbb{H} are the quaternions. If \mathcal{B} is given the hyperkähler structure induced by \mathbb{H} , then $E_{\text{orb}}(\mathcal{B}/W)$ is the same for all complex structures in the hyperkähler family.*

Proof. All of the spaces involved in the computation of the E -polynomials and fermionic shifts are tensor products of real representations of $C(w)$ with the quaternions. As complex representations, they are therefore isomorphic in all complex structures. \square

7 Evaluation for quotients of $\text{SL}(n)$

The orbifold E -polynomial $E_{\text{orb}}((A \otimes \Lambda)/W)$ described above can in fact be evaluated more or less explicitly for the classical groups. For $\text{SL}(n)$ it is computed by Göttsche [19] and shown to coincide with the Hodge polynomial of the crepant resolution, namely the fiber of the natural sum morphism $\text{Hilb}^n A \rightarrow A$. For $\text{Sp}(n)$ it is computed by Bryan-Donagi-Leung [9] and again shown to agree with the crepant resolution, the Hilbert scheme of points on the K3 surface obtained by resolving $A/\pm 1$. (These papers assume A is an abelian surface, but their computations work for any 2-dimensional connected abelian algebraic group.) One can also obtain formulas for the Spin groups and for $\text{SO}(2n)$, but these are more cumbersome.

We present here a computation, based on that of Göttsche [19] and Göttsche-Soergel [20], for $E_{\text{orb}}((A \otimes \Lambda)/W)$, where G is any quotient of $\text{SL}(n)$, and A is any connected abelian algebraic group.

So let l and m be positive integers with $lm = n$, let $G = \text{SL}(n)/\mathbb{Z}_m$, and let Λ be the coweight lattice. The dual group is then $\hat{G} = \text{SL}(n)/\mathbb{Z}_l$, and the Weyl group is the symmetric group S_n . The conjugacy class of any $\sigma \in S_n$ is determined by the partition α of n consisting of the lengths of its cycles: say $n = \sum_j i_j$. Let α_i be the number of cycles of length i , so that $n = \sum_i i\alpha_i$ as well. Let $g = g(\alpha)$ be the greatest common divisor of those i for which $\alpha_i \neq 0$, and let $|\alpha| = \sum \alpha_i$ be the total number of cycles. Recall also that d is the number of $\text{U}(1)$ factors appearing in the decomposition $A = \mathbb{R}^c \times \text{U}(1)^d$.

(7.1) Theorem. *Let A , Λ , and W be as above. Then as a polynomial in u and v ,*

$$E_{\text{orb}}\left(\frac{A \otimes \Lambda}{S_n}\right) = \frac{1}{E(A)} \sum_{\alpha \in P(n)} \tau_{l,m}^{g(\alpha),d} (uv)^{n-|\alpha|} \prod_i E(\text{Sym}^{\alpha_i} A),$$

where $P(n)$ is the set of all partitions $n = \sum_i i\alpha_i$, and

$$\tau_{l,m}^{g,d} = \#\{(r, s) \in \mathbb{Z}_g^d \times \hat{\mathbb{Z}}_g^d \mid (m, g)r = 0 = (l, g)s, \langle r, s \rangle = 1\}.$$

The E -polynomials $E(\text{Sym}^a A)$ can easily be computed as $\bar{E}_{S_a}(A^a)$. See Göttsche-Soergel [20] for a convenient formula when A is an abelian surface.

Proof. Identify the coweight lattice of $\text{GL}(n)$, as an S_n -module, with \mathbb{Z}^n . Then the coweight lattice of $\text{SL}(n)$ is the kernel of the sum map $\mathbb{Z}^n \rightarrow \mathbb{Z}$; the coweight lattice of $\text{GL}(n)/\mathbb{Z}_m$ is

the lattice generated by \mathbb{Z}^n and the point whose coordinates are all $1/m$; and the coweight lattice of G , namely Λ , is the kernel of the latter lattice under the sum map.

Therefore, if K is the kernel of the sum map $A^n \rightarrow A$, then as an S_n -module, $A \otimes \Lambda = K/A[m]$, where $A[m] \cong \mathbb{Z}_m^d$ denotes the division points of order m in A , acting diagonally on A^n and hence on K . The quotient $(A \otimes \Lambda)/S_n$ can then be identified with $K/(A[m] \times S_n)$. By Theorem (5.1), the orbifold E -polynomial can equally well be computed from this quotient.

So fix $\sigma \in S_n$ and $a \in A[m]$. Also, let k be the order of a in $A[m]$, so that $k|m$. To be fixed by the action of $\sigma \times a$, an n -tuple in A^n must consist of cycles of the form

$$(7.2) \quad (x + a, x + 2a, x + 3a, \dots, x + ia)$$

with $x + ia = x$, whose number and size are given by α . Hence, for $(A^n)^{\sigma \times a}$ to be nonempty, it is necessary and sufficient that $k|i$ for every i such that $\alpha_i \neq 0$, and hence that $k|g$. Furthermore, if this is satisfied, then $(A^n)^{\sigma \times a}$ can be identified with $A^{|\alpha|}$ by choosing the first element of each cycle.

However, the intersection of $(A^n)^{\sigma \times a}$ with K may not be connected. Indeed, the sum map $A^n \rightarrow A$ restricts to $(A^n)^{\sigma \times a} = A^{|\alpha|}$ as $(x_j) \mapsto qa + gf_\alpha(x_j)$, where $f_\alpha : A^{|\alpha|} \rightarrow A$ is

$$f_\alpha(x_j) = \sum_j \frac{i_j}{g} x_j,$$

and q is a constant, either 0 or $k/2$, depending on a straightforward parity condition. Consequently, $(x_j) \in (A^n)^{\sigma \times a}$ belongs to K if and only if $f_\alpha(x_j)$ is a g th root of $-qa$. Such g th roots, of course, form a coset of $A[g]$, which partitions $K^{\sigma \times a}$ into g^d disjoint subvarieties, each isomorphic to $K_\alpha = \ker f_\alpha$.

These subvarieties are actually connected, and so constitute the components of $K^{\sigma \times a}$. Indeed, since the i_j/g are coprime, there exist $c_j \in \mathbb{Z}$ such that $\sum_j c_j i_j/g = 1$. The map $A \rightarrow A^{|\alpha|}$ given by $x \mapsto (c_j x)$ is then a right inverse for f_α (cf. [20]), so that $K_\alpha \times A \cong A^{|\alpha|}$.

In fact, more is true. Let Γ be the subgroup $\prod_i S_{\alpha_i} \subset C(\sigma)$ interchanging the cycles of the same length. Then Γ acts on $A^{|\alpha|}$ preserving K_α , and the action commutes with the isomorphism above, if the second factor of $K_\alpha \times A$ is given the trivial Γ -action. Hence

$$(7.3) \quad E_\Gamma(K_\alpha) = \frac{E_\Gamma(A^{|\alpha|})}{E(A)}.$$

The action of the full centralizer $C(\sigma \times a) = C(\sigma) \times A[m]$ does not preserve the components K_α of $K^{\sigma \times a}$, however. First consider the action of $A[m]$. The addition of $b \in A[m]$ to each x_j changes $f_\alpha(x_j)$ to $f_\alpha(x_j + b) = f_\alpha(x_j) + (n/g)b$. So the components of $K^{\sigma \times a}/A[m]$ are indexed by $A[g]/(n/g)A[m] \cong A[(l, g)]$, where $l = n/m$.

As for the action of $C(\sigma)$, it is convenient to break it into two pieces. Let $\Sigma \subset C(\sigma)$ be the normal subgroup $\prod_j \mathbb{Z}_{i_j}$ of $C(\sigma)$, acting cyclically on the cycles of σ . Note that $C(\sigma)/\Sigma$ is the group Γ mentioned before. Since every element of $K^{\sigma \times a}$ has cycles of the form shown in (7.2), the generator of $\mathbb{Z}_{i_j} \subset \Sigma$ acts by adding a to x_j . Since the i_j/g are coprime, the action of Σ can thus add any multiple of a to $f_\alpha(x_j)$. Hence the components of $K^{\sigma \times a}/(A[m] \times \Sigma)$ are indexed by the quotient $A[g]/((n/g)A[m] + \langle a \rangle)$.

Counting the number of elements in this finite abelian group is facilitated by passing to the dual: $A[g] \cong \mathbb{Z}_g^d$ is dual to $\hat{A}[g] = \text{Hom}(A[g], \text{U}(1))$, and so this quotient is dual to the subgroup of $\hat{A}[(l, g)] \subset \hat{A}[g]$ evaluating to 1 on a . Summing over $a \in A[(m, g)]$, we find that the total number of components of

$$\bigcup_{a \in A[(m, g)]} \frac{K^{\sigma \times a}}{A[m] \times \Sigma}$$

is given by the number $\tau_{l, m}^{g, d}$ defined in the statement. Clearly $\tau_{l, m}^{g, d}$ is symmetric in l and m , in agreement with Theorem (6.1). When $(l, g) = 1$ it equals $(m, g)^d$, which is then just g^d since $g|lm$.

The part of $\Sigma \times A[m]$ which preserves K_α acts merely by translations, so dividing by it has no effect on the E -polynomial. Hence

$$\bar{E}_{C(\sigma) \times A[m]} \left(\bigcup_a K^{\sigma \times a} \right) = \tau_{l, m}^{g(\alpha), d} \bar{E}_\Gamma(K_\alpha).$$

The action of Γ , on the other hand, permutes the cycles having the same length. The quotient $A^{|\alpha|}/\Gamma$ is therefore $\prod_i \text{Sym}^{\alpha_i} A$, so according to (7.3),

$$\bar{E}_\Gamma(K_\alpha) = \frac{\prod_i E(\text{Sym}^{\alpha_i} A)}{E(A)}.$$

It remains only to compute the fermionic shifts. But these are easily seen to be independent of a , and in the case $a = 0$ are computed by Göttsche [19] and Göttsche-Soergel [20] to be just $n - |\alpha|$. \square

8 Some final remarks

(8.1) Analogy with the Greene-Plesser construction. In the case where G is semisimple, simply connected and simply laced, that is, of type A , D , or E , then the Langlands dual \hat{G} is simply $G/Z(G)$, and $\hat{H} = H/Z(G)$. Consequently, the spaces appearing in the second columns of the diagrams of §§2 and 3 — that is, the mirrors of the first columns — are quotients of those in the first columns by the finite abelian group $Z(G)^d$. This is eerily reminiscent of the Greene-Plesser construction [21] of the mirror of the Fermat quintic Q as a quotient of Q by the finite abelian group \mathbb{Z}_5^4 . Perhaps this analogy could be expanded to include other groups. Conceivably these spaces could even be realized in a suitable way as subvarieties of dual toric varieties.

(8.2) Topologically non-trivial bundles. This paper deals with only one component of the moduli spaces it employs. An obvious question is how much carries over to other components. Further, one could ask the same question about spaces of bundles which are topologically non-trivial. This would undoubtedly be more difficult. In particular, the work of Borel-Friedman-Morgan [8] would come into play.

(8.3) Hilbert schemes of points. When $G = \mathrm{SL}(n)$, the quotient $(A \otimes \Lambda)/W$ has a crepant resolution, namely the identity fiber $K_{n-1}A$ of the natural map $\mathrm{Hilb}^n A \rightarrow A$. It does not follow from the results of this paper, but it seems extremely plausible, that for $m|n$

$$E_{\mathrm{orb}}(K_{n-1}A/\mathbb{Z}_m^d) = E_{\mathrm{orb}}((A \otimes \Lambda)/(W \times \mathbb{Z}_m^d)).$$

After all, the left-hand side is a partial crepant resolution of the right-hand side. This should be straightforward to check, following Göttsche [19].

(8.4) Equivalence of derived categories. In 1994, Kontsevich proposed an extension of mirror symmetry, the so-called “homological mirror symmetry” [30]. For a mirror pair M, \hat{M} of Calabi-Yau 3-folds, it is supposed to identify two derived categories: that of coherent sheaves on M , and that of the Fukaya category on \hat{M} . In even dimensions, we expect mirror symmetry to identify the A-model of M with the A-model, not the B-model, of \hat{M} . Hence for our examples, we would expect the derived categories of coherent sheaves to be isomorphic for $M(G)$ and $M(\hat{G})$, and likewise for the Fukaya categories.

Even the definition of the Fukaya category remains somewhat mysterious, so there is no hope of proving this equivalence. One might wonder whether hyperkähler rotation transforms the Fukaya category into the category of coherent sheaves, but this is pure speculation.

However, one equivalence, that of derived categories of coherent orbifold sheaves on $\tilde{M}_{\mathrm{Dol}}(G)$ and $\tilde{M}_{\mathrm{Dol}}(\hat{G})$, is easy. Or rather, it follows immediately from the work of Mukai [35]. He showed that the derived categories of coherent sheaves on an abelian variety and its dual are equivalent. The equivalence is defined by pulling back to the product, tensoring by the Poincaré line bundle and pushing forward to the other factor. This can be done for $C \otimes \Lambda$ over the base \mathfrak{h} , and it can be done W -equivariantly, inducing an equivalence of derived categories for sheaves with W -action. The same works for the spaces of §3. Note, however, that even for $G = \mathrm{SL}(2)$ the Poincaré line bundle does not descend to a bona fide sheaf on the quotient by W . Using orbifold sheaves therefore seems necessary.

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